Modified Fractional Curl Operator

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Abstract—Applying fractional calculus in the field of electromagnetics shows significant results. The fractionalization of the conventional curl operator leads to have additional solutions to an electromagnetic problem. This work restudies the concept of the fractional curl operator considering fractional time derivatives in Maxwell’s curl equations. In that sense, a general scheme for the wave loss term is introduced and the degree of freedom of the system is affected through imposing the new fractional parameters. The conventional case is recovered by setting all fractional derivatives to unity.

Index Terms—fractional calculus, fractional curl operators, Maxwell equations.

I. INTRODUCTION

Recently, attention has been given to applying fractional calculus in electromagnetic to study the intermediate behavior between the integer-order responses which reflects better understanding, promising results and new ideas. Fractional solutions for the Helmholtz’s equation were discussed in [1-2], the analysis for a homogeneous space which means the geometry of the problem is unbounded was discussed in [1] while different geometries with parallel plane interfaces were considered in [2]. Dedication to studying fractional electromagnetics in fractal spaces is found in [3-4], whereas Maxwell’s equations form in fractal spaces was given in [3], the cylindrical wave equation in fractional dimensional space was solved in [4]. One of the main advantages for the investigation of the fractional-order in the electromagnetics is to expand the fundamentals into a new domain, such as the concept of fractional-order in the electromagnetics is to expand the fundamentals into a new domain, such as the concept of fractional-order Smith chart [5-6]. It was proved through these papers that the conventional Smith chart is a special case, and the new chart becomes a 3D shape with many interesting fundamentals that can be applied in many applications such as impedance matching [6].

The basic definitions of the fractional integral ($I^\alpha$) and fractional derivative ($D^\alpha$) of a function $f(t)$ due to Caputo [7-8], useful in many engineering applications, are given by

\[ I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha \in \mathbb{R}^+, t > 0 \]  

\[ D^\alpha f(t) = I^{m-\alpha} (D^m f(t)), \quad m - 1 < \alpha \leq m \]  

where $m \in \mathbb{N}^+$. The Laplace transform of Caputo fractional derivative is given as

\[ L[D^\alpha f(t)] = S^\alpha F(S) - \sum_{k=0}^{m-1} \left( \frac{d^k}{dt^k} f(t) \right) \bigg|_{t=0} S^{\alpha-k} \]  

Unlike the conventional derivative, as depicted in (1), to calculate the fractional derivative of a certain function $f(t)$, all the history of this function is required. Accordingly, the fractional-order modeling has advantages for many long-memory dependence processes better than the conventional integer-order models. Moreover, the extra parameters (fractional-orders) increase the degrees of freedom and the design flexibility for improving the response of a system.

During the last three decades, various systems analysis based on the fractional-order calculus were introduced mathematically and verified experimentally in different fields such as electromagnetics, electric circuit theory, mechanical system control, and bioengineering [9-11]. In the field of fractional electromagnetics considering fractional-order time derivatives, some work [12-13] found that the integer-order diffusion equations cannot accurately describe the diffusion phenomenon whereas the fractional diffusion equations led to proper and realistic description of such processes. Another work [14] studied the effect of fractional-order derivatives with respect to time on the generalized analysis and fundamentals of the rectangular waveguide. The effect of the fractional-order parameters on the waveguide properties is shown. Some of these properties are cutoff frequency, attenuation and intrinsic impedance that are shown to be complex in value.

Nader Engheta initiated an important concept called the fractional curl operator [15] by introducing the tools of fractional calculus into the theory of electromagnetism; he termed this special area of electromagnetics as fractional paradigm in electromagnetic theory. Consequently, many fractional-order electromagnetic research papers were published during the last two decades [16-17]. During mathematical treatment in fractional paradigm, in order to solve a general electromagnetic problem, the canonical solutions of the problem are derived at the beginning then a fractional differential operator that can transform one canonical solution into the other is derived. Using this operator, other intermediate solutions between the two canonical ones may be obtained.

In this work, this concept of fractional curl operator is restudied upon the consideration of fractional-order time derivatives in Maxwell’s curl equations. In this sense, an extra fractional parameter is imposed on the concept of fractional curl operator leading to have an additional degree of freedom to control its characteristics. Moreover, this added parameter introduces a power loss term that can model the frequency-dependent losses in case all resistive elements in the electromagnetic problem under study are neglected.

The organization of this paper is as follows: Section II shortly restudies Maxwell’s equations taking into account a fractional-order time derivatives. The fractional curl operator formulas based on the modified form of Maxwell’s equations are introduced in section III applying on a case of a one

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dimensional wave propagation. Finally, conclusions are found in section IV.

II. MAXWELL’S EQUATIONS IN THE FRACTIONAL-ORDER FORM

Conventionally, the displacement current density \( j_d \) due to electric field intensity \( E \) in a dielectric with a permittivity \( \varepsilon \) is given as:

\[
j_d = \varepsilon \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t}
\]

where \( P \) is the polarization vector of the medium. This displacement current density \( j_d \) with the free current density \( j_f \) are related to the magnetic field density \( H \) through the Maxwell curl equation

\[
\nabla \times H = j_f + j_d
\]

As depicted in (3), the displacement current is related to the rate of change of the electric displacement field and it exists if there is a time-varying electric field. This current is modeled as the current passing through an integer-order capacitor that means lossless term which is impractical. Recently, the conventional capacitor is considered as a special case of the fractional-order one [19] where the phase difference between the voltage and current is equal to \( \frac{\pi}{2} \), leading to have a lossy power term that is frequency dependent which agrees with the Coilcraft experimental work presented in [20], and hence there is a need to study fractional-order displacement current. Therefore, it is more reasonable to reinvent the Maxwell’s equations in the fractional-time order derivatives to extract more fundamentals that exist in the fractional-order solutions which will automatically tend to the conventional solution as the entire fractional orders set to unity. According to Faraday’s law for induction and with a general fractional-order derivative \( \alpha \) we have the first Maxwell’s curl equation

\[
\nabla \times E = -\mu \frac{\partial^\alpha}{\partial t^\alpha} H
\]

Similarly, for some general fractional-order derivative \( \beta \), the 2nd Maxwell curl equation can be written as:

\[
\nabla \times H = -\varepsilon \frac{\partial^\beta}{\partial t^\beta} H
\]

Consider an electromagnetic plane wave propagating in the \( z \)-direction in a source-free medium with parameters \( \mu, \varepsilon, \alpha \) and \( \beta \) then the associated Maxwell equations for the phasor representation of the electromagnetic fields \( E_z \), \( H_z \) assuming harmonic time dependence \( e^{j\omega t} \) are as follows:

\[
\nabla \times E_z = -(j\omega) \mu H_z
\]

\[
\nabla \times H_z = (j\omega) \varepsilon E_z
\]

\[
\nabla \cdot E_z = 0, \nabla \cdot H_z = 0
\]

Solving for \( E_z \) assuming that the electric field is polarized only along the \( x \)-direction, i.e.;

\[
E_z(x) = E_0 e^{y^2} \xi
\]

where \( \gamma \) is the propagation constant, it is easy to mathematically deduce that \( \gamma \) is complex in value, i.e. \( \gamma = \sigma + j\delta \) for the two real numbers \( \sigma \) and \( \delta \) that is

\[
E_z(x) = e^{\sigma x} e^{j\delta x}
\]

with two solutions available

\[
\sigma_{1,2} = \pm \sqrt{\mu\varepsilon (\omega)}^{\frac{\alpha+\beta}{2}} \cos \left( \frac{(\alpha + \beta)\pi}{4} \right)
\]

Then \( \gamma \) has two distinct complex values \( \gamma_1 \) and \( \gamma_2 \) depending on the propagation direction. Conventionally \( (\alpha = \beta = 1) \), these two solutions will be reduced to \( \pm j\omega \sqrt{\mu\varepsilon} \) as known. It is noticed that when \( 0 < \alpha + \beta < 2 \) (passive media), \( \sigma_1 \) and \( \delta_1 \) are positive confirming that the propagation is along the negative-\( z \) direction and hence a decaying wave is obtained. Conversely, for \( 2 < \alpha + \beta < 4 \) (reflects that the media is an active source of energy), \( \sigma_2 \) is negative, however \( \delta_1 \) is positive confirming also that the propagation is along the negative-\( z \) direction with an exponential growing-up wave is obtained which is logic since the media provide the wave an extra power but not acceptable practically due to the absence of such media. Accordingly, the only accepted region for both values of \( \gamma \) is \( 0 < \alpha + \beta < 2 \) leading to only consider \( 0 < \alpha, \beta < 1 \) which represent a lossy(decaying) wave but the most important issue here is that this decaying is frequency dependent as approved by Coilcraft report [20]. This means that imposing fractional parameters gives a general study taking the advantage of having a more degree of freedom to control the attenuation constant through the value of the fractional parameters. The corresponding magnetic field vector can be obtained through the curl equation

\[
\nabla \times E_z = -(j\omega) \mu H_z
\]

This leads to

\[
H_z = E_0 \sqrt{\frac{\varepsilon}{\mu}} (j\omega)^{\frac{\beta-\alpha}{2}} e^{jyz} \hat{y}
\]

Then the intrinsic impedance \( \eta_f = \frac{E_z}{H_z} \) is given by

\[
\eta_f = \frac{\sqrt{\mu}}{\sqrt{\varepsilon}} (j\omega)^{\frac{\alpha-\beta}{2}} \left( \cos \left( \frac{(\alpha - \beta)\pi}{4} \right) + j \sin \left( \frac{(\alpha - \beta)\pi}{4} \right) \right)
\]

which is a complex value but when \( \alpha = \beta \) (rate of change in (5) and (6) are of equal orders) the conventional real-valued expression \( \eta_f = \sqrt{\mu/\varepsilon} \) is retrieved. Figure 1 shows the real and imaginary parts of the fractional-order intrinsic impedance when \( \alpha < \beta \) and \( \alpha > \beta \) for \( \sqrt{\mu/\varepsilon} = 120 \Omega \). It is clear for the first case that the real values becomes very low when \( \alpha < \beta \) for all frequencies and increases up to 120\( \Omega \) when \( \alpha = \beta \) for all frequencies. The imaginary values fluctuate within a small range. However, both the real and imaginary values increase so much as long as \( \alpha > \beta \) up to thousands.

In the following section the concept of fractional curl operator [15] is discussed based on the time domain fractional derivative.
\[
E_{fd} = \left[ \frac{1}{(ik_f)^\rho} \right] \frac{\nabla \times E}{E} \tag{14a}
\]
\[
\eta_f H_{fd} = \left[ \frac{1}{(ik_f)^\rho} \right] \frac{\nabla \times (\eta_f H)}{H} \tag{14b}
\]

with \( ik_f = (i\omega)^{\alpha+\beta} \sqrt{\mu/\varepsilon} \). It should be noted that \( ik_f \) reduces to
\( i\omega\sqrt{\mu/\varepsilon} = ik \) in the conventional case, where \( k \) is the wave number. When \( \rho = 0 \), one finds that \( E_{fd} = E \) and \( \eta_f H_{fd} = \eta_f H \), which is the original set of solutions satisfying (7) and for \( \rho = 1 \) one gets
\[
E_{fd} = \frac{1}{ik_f} \nabla \times E = \eta_f H \tag{15a}
\]
\[
\eta_f H_{fd} = \frac{1}{ik_f} \nabla \times (\eta_f H) = -E \tag{15b}
\]

which are the dual fields of the original solutions to the Maxwell equations. Accordingly, for all values of \( \rho \) between zero and one (14) give new set of solutions that can be effectively intermediate solutions between the original and the dual fields. These solutions are also called the fractional dual fields as expressed with the subscript \( fd \) for them. By setting all fractional parameters to one the conventional case of the fractional curl operator is retrieved [15].

Now, the fractional dual solution concept in the modified form (14) is applied on the above example of a travelling plane wave. As derived above the field solutions can be rewritten as
\[
E_{fd}(z) = E_0 e^{i\gamma z} \hat{x} \tag{16a}
\]
\[
\eta_f H_{fd}(z) = E_0 e^{i\gamma z} \hat{y} \tag{16b}
\]

Now, let us have this important remark, one can easily shows that the electromagnetic fields in a source-free medium satisfy
\[
\frac{1}{ik_f} \nabla \times \frac{\nabla \times E}{E} = -\eta_f H \tag{17a}
\]
\[
\frac{1}{ik_f} \nabla \times (\eta_f H) = E \tag{17b}
\]

It can easily check that the field solutions in (16) satisfy
\[
(-\hat{z}) \times E = -\eta_f H \tag{18a}
\]
\[
(-\hat{z}) \times \eta_f H = E \tag{18b}
\]

and hence the operator \( \frac{1}{ik_f} (\nabla \times ) \) in this problem is equivalent to the operator \( (-\hat{z}) \times \), this indicates that the fractional operator \( \frac{1}{ik_f}^{\alpha+\beta} (\nabla \times )^\rho \) is also equivalent to \( (-\hat{z}) \times )^\rho \) in this problem. This important remark helps us to find easily the fractional dual solutions (14) of the considered problem as depicted in the technique of the fractionalization of the curl operator [15]. To find the fractional dual solutions for this problem the field vectors should be written in terms of eigenvectors of the cross product \(( -\hat{z}) \times \) and accordingly by using (14) the fractional dual fields are obtained as:
\[
E_{fd} = E_0 \left[ \cos \left( \frac{\mu \omega z}{2} \right) \hat{x} + \sin \left( \frac{\mu \omega z}{2} \right) \hat{y} \right] e^{\gamma z} \tag{19a}
\]
\[
\eta_f H_{fd} = E_0 \left[ -\sin \left( \frac{\mu \omega z}{2} \right) \hat{x} + \cos \left( \frac{\mu \omega z}{2} \right) \hat{y} \right] e^{\gamma z} \tag{19b}
\]

It may be noted that for \( \rho = 0 \) the above set of expressions yield \(( E_0, \eta_f H_0 )\) and for \( \rho = 1 \) it yields \(( \eta_f H_0, -E_0 )\) which are the solutions of the duality theorem. For \( 0 < \rho < 1 \), the fields can be regarded as fractional dual fields between the original and dual to the original fields of the plane wave propagating in the z-direction. Also, it is observed that from (19) the fractional

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III. FRACTIONAL CURL OPERATOR

Based on the concept of fractional curl operator [15] presented by Engheta, several works had been published with many interesting results [16-17]. Conventionally, duality theorem [21] in electromagnetics states that if the set of electromagnetic fields \(( E, \eta H )\) is a solution to an electromagnetic problem then its dual set \(( \eta H, -E )\) is also a solution set for the same problem where \( \eta \) is the intrinsic impedance of that medium. The main idea of the fractional curl operator is to transform between the canonical solutions of a general electromagnetic problem by this operator and investigate other intermediate solutions between the two canonical solutions. This was achieved by fractionalizing the conventional curl operator \( (\nabla \times ) \) where the new fractionalized operator, denoted by \( (\nabla \times )^\rho \) with the fractional parameter \( \rho \), can be used to obtain the intermediate solutions between the canonical ones of an electromagnetic problem [15]. Several applications were introduced based on this idea presented in [16-17]. All the previous work was based on the conventional Maxwell’s integer-order equations.

Now, based on the modified form of fractional-order time derivative Maxwell’s equations (5-6), the fractional curl operator is restudied giving its modified formulas. Consider again the source-free Maxwell equations (7), these equations can be reorganized and manipulated according to the technique of fractionalizing the curl operator given in [15] to find the new set of solutions in the form

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Fig. 1. The Fractional Intrinsic Impedance Versus \( \omega \) And \( \alpha - \beta \) When \( \sqrt{\mu/\varepsilon} = 120 \) When (A) \( < \beta \), And (B) \( \alpha > \beta \)
dual fields represent a plane wave propagating in the same direction as the original wave, however its transverse fields have been rotated by an angle $\frac{\rho \pi}{2}$. By substituting $\gamma$, one gets the complete forms of the fractional dual fields including all fractional parameters:

$$E_{fd} = E_0 \left[ \cos \left( \frac{\rho \pi}{2} \right) \hat{x} + \sin \left( \frac{\rho \pi}{2} \right) \hat{y} \right] e^{-\left( \sqrt{\mu \epsilon} \frac{\alpha + \beta}{\rho \pi} \right) x} e^{\left( \sqrt{\mu \epsilon} \frac{\alpha + \beta}{\rho \pi} \right) z}$$  \hspace{1cm} (20a)

$$\eta \gamma H_{fd} = E_0 \left[ -\sin \left( \frac{\rho \pi}{2} \right) \hat{x} + \cos \left( \frac{\rho \pi}{2} \right) \hat{y} \right] e^{-\left( \sqrt{\mu \epsilon} \frac{\alpha + \beta}{\rho \pi} \right) x} e^{\left( \sqrt{\mu \epsilon} \frac{\alpha + \beta}{\rho \pi} \right) z}$$  \hspace{1cm} (20b)

The last form indicates that taking into account fractional time derivatives a general loss term is shown to indicate losses observed upon wave propagation in free-source mediums and additional field solutions intermediate to the dual ones can be obtained by considering fractional curl operators.

Certainly, it’s more convenient to have a practical example from the conventional case to see the effect of imposing fractional parameters on its performance. The rectangular waveguide case is selected and studied as a point of future work.

IV. CONCLUSIONS

Maxwell’s curl equations are reconsidered replacing the integer-order time derivatives by fractional ones. By applying the modified form on the operation of an electromagnetic plane wave propagating in a source-free medium a general expression for the loss term of the wave is introduced which recovers the conventional case by setting all fractional derivatives to unity. Afterwards, the modified formulas of the fractional curl operator taking into account the modified form of Maxwell’s curl equations are introduced. Following this, an additional degree of freedom to control the characteristics of the fractional dual solutions is introduced due to the extra fractional parameter. Applying this work on a practical example of a rectangular waveguide is a point of future work expected to introduce significant results.

REFERENCES


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